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Some new topological cardinal inequalities

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Abstract

In this paper we make use of the Pol–Šapirovič's technique to prove several cardinal inequalities, which generalize other well-known inequalities.

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1. Introduction

Among the best known theorems concerning cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. In [5], Hodel classified the bounds on $|X|$ in two categories namely *easy* and *difficult* to prove. For instance, the following inequalities are in the difficult category:

- (1) (Arhangel'skiĭ) If X is a T_2 -space then $|X| \leq 2^{L(X)\psi(X)t(X)}$.
- (2) (Hajnal–Juhász) If X is a T_2 -space then $|X| \leq 2^{c(X)\chi(X)}$.
- (3) (Charlesworth) If X is a T_1 -space, $|X| \leq psw(X)^{L(X)\psi(X)}$.
- (4) (Bell–Ginsburg–Woods) If $X \in \mathcal{T}_4$, $|X| \leq 2^{wL(X)\chi(X)}$.

In [4] Fedeli introduces three cardinal functions, two of these are lc and aql , and he uses the language of elementary submodels to prove:

- (5) If X is a T_2 -space, $|X| \leq 2^{aql(X)\psi_c(X)t(X)}$.
- (6) If X is a T_2 -space, $|X| \leq 2^{lc(X)\pi\chi(X)\psi_c(X)}$.

These inequalities improve (1) and (2), respectively. Other generalizations of (1) and (2) have also been proved by Shu-Hao [10]:

- (7) If X is a T_2 -space, $|X| \leq 2^{ql(X)\psi_c(X)t(X)}$.

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(8) If X is a T_2 -space, $|X| \leq 2^{c(X)\pi\chi(X)\psi_c(X)}$.

However the inequalities (7) and (8) are respectively consequences of (5) and (6) (see [4]).

In this paper we will establish several cardinal inequalities which generalize the results mentioned above. Later we give an example to show that our inequalities can be better estimations than the previously mentioned ones.

2. Notation and terminology

We refer the reader to [5,4,7] for definitions and terminology not explicitly given here. Let L , c , wl , psw , χ , ψ , $\pi\chi$ and t denote the following standard cardinal functions: Lindelöf degree, cellularity, weak Lindelöf degree, point separating weight, character, pseudocharacter, π -character and tightness respectively.

The following cardinal functions are due to Fedeli [4].

Definition 1. Let X be a topological space:

- (a) $ac(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open collection \mathcal{U} in X , there is a $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$, with $\bigcup \mathcal{U} \subseteq S \cup \bigcup \{V : V \in \mathcal{V}\}$.
- (b) $lc(X)$ is the smallest infinite cardinal κ such that there is a closed subset F of X such that $|F| \leq 2^\kappa$ and for every open collection \mathcal{U} in X , there is a $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$, with $\bigcup \mathcal{U} \subseteq F \cup \bigcup \{V : V \in \mathcal{V}\}$.
- (c) $aql(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open cover \mathcal{U} of X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ with $X = S \cup (\bigcup \mathcal{V})$.

Clearly $ac(X) \leq lc(X) \leq c(X)$ (see [5, Proposition 3.4]), and $aql(X) \leq L(X)$ for every topological space.

Following the definition of aql , we introduce a new cardinal function.

Definition 2. Let X be a topological space:

$qwl(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open cover \mathcal{U} of X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ with $X = S \cup \bigcup \mathcal{V}$.

Clearly $qwl(X) \leq wl(X)$, $qwl(X) \leq aql(X)$ and $qwl(X) \leq d(X)$ for every topological space X .

Finally, Shu-Hao [10], introduced the following cardinal invariant:

Definition 3. Let X be a topological space:

$ql(X)$ is the smallest infinite cardinal κ such that there is a subset A of X such that A is κ -quasi-dense, where $A \subseteq X$ is κ -quasi-dense if $|A| \leq 2^\kappa$ and for every open cover \mathcal{U} of X there are $B \in [A]^{\leq \kappa}$ and $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $X = \overline{B} \cup \bigcup \mathcal{V}$.

Clearly $ql(X) \leq d(X)$, and $wL \leq ql(X) \leq L(X)$ for every topological space.

Definition 4. Let X be a Hausdorff space. The closed pseudocharacter of X , denoted $\psi_c(X)$, is the smallest infinite cardinal κ such that for every $x \in X$ there is a collection \mathcal{U}_x of open neighborhoods of x such that $\bigcap \{\overline{U} : U \in \mathcal{U}_x\} = \{x\}$ and $|\mathcal{U}_x| \leq \kappa$ (see [7]).

Note that $\psi_c(X) = \psi(X)$ if X is a T_3 space.

3. Main results

The following remarkable result is due to Arhangel'skiĭ [1]:

(a) If X is a T_1 space, such that: (i) $L(X)t(X) \leq \kappa$; (ii) $\psi(X) \leq 2^\kappa$, and (iii) for all $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$. Then $|X| \leq 2^\kappa$.

Since $aql(X) \leq L(X)$ and $ql(X) \leq L(X)$ for every topological space X , it is natural to ask if L can be replaced by aql or ql in the inequality (a). Ramírez-Páramo [9] gives an affirmative answer for aql . The next theorem gives an affirmative answer for ql .

Theorem 5. If X is a T_1 -space, such that: (i) $ql(X)t(X) \leq \kappa$; (ii) $\psi(X) \leq 2^\kappa$, and (iii) if $A \in [X]^{\leq 2^\kappa}$, then $|\bar{A}| \leq 2^\kappa$. Then $|X| \leq 2^\kappa$.

Proof. Let S be an element of $[X]^{\leq 2^\kappa}$ witnessing that $ql(X) \leq \kappa$. For each $x \in X$, let \mathcal{B}_x a pseudobase of x in X such that $|\mathcal{B}_x| \leq 2^\kappa$.

Construct an increasing sequence $\{A_\alpha: \alpha \in \kappa^+\}$ of closed sets in X and a sequence $\{\mathcal{V}_\alpha: \alpha \in \kappa^+\}$ of open collections in X such that

- (1) $|A_\alpha| \leq 2^\kappa$, $0 \leq \alpha < \kappa^+$;
- (2) $\mathcal{V}_\alpha = \bigcup \{\mathcal{B}_x: x \in cl_X(\bigcup_{\beta < \alpha} A_\beta)\}$, $0 < \alpha < \kappa^+$;
- (3) if $\mathcal{U} \subseteq \bigcup \{\mathcal{B}_x: x \in cl_X(\bigcup_{\beta < \alpha} A_\beta)\}$ with $|\mathcal{U}| \leq \kappa$, and $B \in [S]^{\leq \kappa}$, such that $X - (\bar{B} \cup \bigcup \mathcal{U}) \neq \emptyset$, then $A_\alpha - (\bar{B} \cup \bigcup \mathcal{U}) \neq \emptyset$.

Let $A = \bigcup \{A_\alpha: \alpha \in \kappa^+\}$ and note that, as $t(X) \leq \kappa$, A is closed in X ; moreover, clearly $|\bar{S} \cup A| \leq 2^\kappa$. Let $\mathcal{V} = \bigcup \{\mathcal{V}_\alpha: \alpha \in \kappa^+\}$. The proof is complete if $X = \bar{S} \cup A$. Suppose not and let $p \in X - (\bar{S} \cup A)$. For each $x \in A$ choose $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. Then $\mathcal{W} = \{V_x: x \in A\}$ together with $\{X - A\}$ covers X ; hence, there exist $B \subseteq [S]^{\leq \kappa}$ and $\mathcal{U} \in [\mathcal{W}]^{\leq \kappa}$ such that $X = \bar{B} \cup \bigcup \mathcal{U} \cup (X - A)$. Clearly $A \subseteq \bar{B} \cup \bigcup \mathcal{U}$ and $p \notin \bar{B} \cup \bigcup \mathcal{U}$; moreover, since $|\mathcal{U}| \leq \kappa$ and κ^+ is regular, there is $\alpha \in \kappa^+$ such that $\mathcal{U} \in [\mathcal{V}_\alpha]^{\leq \kappa}$ and $X - (\bar{B} \cup \bigcup \mathcal{U}) \neq \emptyset$; hence by (3), $A_\alpha - (\bar{B} \cup \bigcup \mathcal{U}) \neq \emptyset$. Since $A_\alpha \subseteq A \subseteq S \cup U$, we reach a contradiction. Thus $X = \bar{S} \cup A$. \square

Now we have the inequality (a), as a consequence of our theorem.

Corollary 6 (Arhangel'skiĭ). If X is a T_1 -space, such that: (i) $L(X)t(X) \leq \kappa$; (ii) $\psi(X) \leq 2^\kappa$, and (iii) for all $A \in [X]^{\leq 2^\kappa}$, $|\bar{A}| \leq 2^\kappa$. Then $|X| \leq 2^\kappa$.

Another consequence of Theorem 5 is Shu-Hao's inequality (see [10]).

Corollary 7. If X is a T_2 -space then $|X| \leq 2^{ql(X)\psi_c(X)t(X)}$.

Proof. Let $\kappa = ql(X)\psi_c(X)t(X)$. It is enough to note that for all $A \in [X]^{\leq 2^\kappa}$, $|\bar{A}| \leq 2^\kappa$. \square

It is easy to prove that $aql(X) \leq ql(X)\psi_c(X)t(X)$ for every Hausdorff space X , however aql is not directly comparable with ql . To see this, consider the following examples.

Example 8. Let $X = D(2^\kappa)$ be the discrete space of cardinality 2^κ ; for some cardinal $\kappa \geq \omega$. Then $aql(X) \leq \kappa$ and $ql(X) > \kappa$.

Example 9. Let X be the Katětov H -closed extension of \mathbb{N} . This space contains a countable dense subset A consisting of isolated points of X such that the subspace $X - A$ is discrete. It is easy to check $ql(X) = d(X) = \omega$. Note that $aql(X) > \omega$. Indeed, suppose $aql(X) = \omega$ and let S be an element of $[X]^{\leq 2^\omega}$ witnessing that $aql(X) = \omega$. Now for each $x \in X - A$ there is U_x an open neighborhood of x , such that $U_x \cap (X - A) = \{x\}$. Let $\mathcal{U} = \{U_x: x \in X - A\} \cup \{\{a\}: a \in A\}$. Clearly \mathcal{U} is an open cover of X . Since $aql(X) = \omega$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq \omega}$, such that $X = S \cup \bigcup \mathcal{V}$. Then $|X| \leq |S| + |\bigcup \mathcal{V}| \leq 2^\omega$. A contradiction. Therefore, $aql(X) > \omega$.

It is well known that $\psi(X) \leq \chi(X)$ for all $X \in \mathcal{T}_1$, Arhangel'skiĭ has asked if the pair $\{L, \psi\}$ bounds $|X|$. This difficult question is discussed in [8] by Juhász. A partial solution, due to Charlesworth [2], states that $|X| \leq psw(X)^{L(X)\psi(X)}$. Of course, it is natural to ask if L can be replaced by aql or ql in the Charlesworth's inequality. At the moment the authors do not know the answer when L is replaced by ql ; however, the answer when L is replaced by aql is yes. To see this we need the following results.

Theorem 10. If X is a T_1 -space, $d(X) \leq psw(X)^{aql(X)}$.

Proof. Let $\kappa = aql(X)$, let S be an element of $[X]^{\leq 2^\kappa}$ witnessing that $aql(X) = \kappa$ and let \mathcal{V} be an separating open cover of X such that for every $x \in X$, $ord(x, \mathcal{V}) \leq \gamma$, where $\gamma = psw(X)$. For each $x \in X$, denote by \mathcal{V}_x the collection of the elements $V \in \mathcal{V}$ such that $x \in V$. Construct a sequence $\{D_\alpha: 0 \leq \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha: 0 < \alpha < \kappa^+\}$ of open collections in X such that

- (1) $|D_\alpha| \leq \gamma^\kappa$, $0 \leq \alpha < \kappa^+$;
- (2) $\mathcal{V}_\alpha = \bigcup \{\mathcal{V}_x: x \in \bigcup_{\beta < \alpha} D_\beta\}$, $0 \leq \alpha < \kappa^+$;
- (3) if U is a union of $\leq \kappa$ elements of \mathcal{V}_α and $X \setminus (S \cup U) \neq \emptyset$, then $D_\alpha \setminus (S \cup U) \neq \emptyset$.

Let $D = \bigcup \{D_\alpha: \alpha \in \kappa^+\}$. It is clear that $|S \cup D| \leq \gamma^\kappa$.

The proof is complete if $X = \overline{S \cup D}$. Suppose that $p \in X - (\overline{S \cup D})$ and let $\mathcal{W} = \{V \in \mathcal{V}: V \cap D \neq \emptyset \text{ and } p \notin V\}$. Clearly $X = \bigcup \mathcal{W} \cup (X \setminus \overline{D})$, hence, since $aql(X) \leq \kappa$, there exists $\mathcal{U} \subseteq \mathcal{W}$ with $|\mathcal{U}| \leq \kappa$ such that $X = S \cup \bigcup \mathcal{U} \cup (X \setminus \overline{D})$. Then $D \subseteq \overline{D} \subseteq S \cup \bigcup \mathcal{U}$ and $p \notin S \cup \bigcup \mathcal{U}$. Let $U = \bigcup \mathcal{U}$. Since $|\mathcal{U}| \leq \kappa$, by regularity of κ^+ , there exists $\alpha_0 \in \kappa^+$ such that U is a union of $\leq \kappa$ elements of \mathcal{V}_{α_0} and $X \setminus (S \cup U) \neq \emptyset$. Hence by (3), $D_{\alpha_0} \setminus (S \cup U) \neq \emptyset$. Since $D_{\alpha_0} \subseteq D \subseteq S \cup \bigcup \mathcal{U}$, we reach a contradiction. \square

It is not difficult to show that the Theorem 10 holds if aql is replaced by ql .

A consequence of Theorem 10 is the following generalization of the Charlesworth's inequality [2].

Corollary 11. *If X is a T_1 -space, $nw(X) \leq psw(X)^{aql(X)}$.*

Proof. Let $\gamma = psw(X)$ and let $\kappa = aql(X)$. Fix S , an element of $[X]^{\leq 2^\kappa}$ witnessing that $aql(X) = \kappa$ and fix \mathcal{V} a separating open cover of X , such that for each $x \in X$, $|\mathcal{B}_x| \leq \gamma$ where \mathcal{B}_x denotes the collection of members of \mathcal{V} containing x . By Theorem 10, $d(X) \leq \gamma^\kappa$. Let D be a dense subset of X with $|D| \leq \gamma^\kappa$. It is clear that $\mathcal{V} = \bigcup \{\mathcal{B}_x: x \in D\}$; hence $|\mathcal{V}| \leq \gamma^\kappa$. Let

$$\mathcal{N} = \left\{ X - \left(S \cup \bigcup \mathcal{U} \right): \mathcal{U} \in [\mathcal{V}]^{\leq \kappa} \right\} \cup \{ \{s\}: s \in S \}.$$

It is clear $|\mathcal{N}| \leq \gamma^\kappa$; moreover \mathcal{N} is a net for X . Indeed, let U an nonempty open set of X and consider $p \in U$. We have two cases:

- (1) $p \in S$. Then $\{p\} \in \mathcal{N}$ and $p \in \{p\} \subseteq U$.
- (2) $p \notin S$. For each $x \in X - \{p\}$, let $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. Denote \mathcal{W} the collection of elements of \mathcal{V} chosen in this way. Then $\mathcal{W} \cup \{U\}$ is an open cover of X . Since $aql(X) = \kappa$, there exists $\mathcal{U} \in [\mathcal{W}]^{\leq \kappa}$ such that $X = S \cup \bigcup \mathcal{U} \cup U$. It is clear that $p \in X - (S \cup \bigcup \mathcal{U}) \subseteq U$.

Therefore \mathcal{N} is a net for X , hence $nw(X) \leq psw(X)^{aql(X)}$. \square

The following result is due to Charlesworth [2].

Lemma 12. *If X is a T_1 -space, then $|X| \leq nw(X)^{\psi(X)}$.*

Proof. Let $\kappa = nw(X)$, $\gamma = \psi(X)$ and let \mathcal{N} be a net for X with $|\mathcal{N}| \leq \kappa$. Then $\{\{x\}: x \in X\} \subseteq \{\bigcap \mathcal{N}': \mathcal{N}' \subseteq \mathcal{N}, |\mathcal{N}'| \leq \gamma\}$. Therefore $|X| \leq \kappa^\gamma$. \square

We are in position to prove a strengthening of the Charlesworth's inequality [2].

Theorem 13. *If X is a T_1 -space, then $|X| \leq psw(X)^{aql(X)\psi(X)}$.*

Proof. By Lemma 12, $|X| \leq nw(X)^{\psi(X)}$, hence by Corollary 11, we have $|X| \leq (psw(X)^{aql(X)})^{\psi(X)} = psw(X)^{aql(X)\psi(X)}$. \square

A consequence of Theorem 13 is the Charlesworth's inequality [2].

Corollary 14. *If X is a T_1 -space, then $|X| \leq \text{psw}(X)^{L(X)\psi(X)}$.*

The following example shows that Theorem 13 can give better estimation than one in Corollary 14.

Example 15. Let $X = D(2^\kappa)$ be the discrete space of cardinality 2^κ , for some cardinal $\kappa \geq \omega$. Note that $\text{psw}(X) = 2^\kappa$, $\psi(X) = \omega$, $L(X) = 2^\kappa$, $\text{acl}(X) \leq \kappa$; hence $\text{psw}(X)^{L(X)\psi(X)} = 2^{2^\kappa} > 2^\kappa = \text{psw}(X)^{\text{acl}(X)\psi(X)} = |X|$.

It is well known that the Hajnal–Juhász’s inequality (1) (in our Introduction), and Šapírovskii’s inequality: (9) For $X \in \mathcal{T}_3$, $|X| \leq \pi \chi(X)^{c(X)\psi(X)}$ are still among the best cardinal inequalities involving the cardinality of X . In [10], Shu-Hao proved: (10) For $X \in \mathcal{T}_2$, $|X| \leq \pi \chi(X)^{c(X)\psi_c(X)}$ which is a common generalization of the two inequalities (2) and (9). Since $\text{ac}(X) \leq \text{lc}(X) \leq c(X)$, it is natural to ask if c can be replaced by ac or lc in the inequality (10). At the moment, the authors do not know the answer for ac ; however, the following theorem shows that c can be replaced by lc in the Shu-Hao’s inequality.

Theorem 16. *If X is a T_2 -space then $|X| \leq \pi \chi(X)^{\text{lc}(X)\psi_c(X)}$.*

Proof. Let $\lambda = \pi \chi(X)$, $\kappa = \text{lc}(X)\psi_c(X)$, and let F be a closed set in X with $|F| \leq 2^\kappa$ and witnessing that $\text{lc}(X) \leq \kappa$. For each $x \in X$, let \mathcal{B}_x a local π -base of x in X such that $|\mathcal{B}_x| \leq \kappa$.

Construct a sequence $\{A_\alpha: \alpha \in \kappa^+\}$ of sets in X and a sequence $\{\mathcal{V}_\alpha: \alpha \in \kappa^+\}$, of open collections in X such that:

- (1) $|A_\alpha| \leq \lambda^\kappa$; $0 \leq \alpha \leq \kappa^+$;
- (2) $\mathcal{V}_\alpha = \bigcup \{\mathcal{B}_x: x \in \bigcup_{\beta < \alpha} A_\beta\}$; $0 < \alpha < \kappa^+$;
- (3) if $\{\mathcal{U}_\gamma: \gamma \in \kappa\} \subseteq [\mathcal{V}_\alpha]^{\leq \kappa}$ and $X \neq (F \cup \bigcup \{\bigcup \{V: V \in \mathcal{U}_\gamma\}: \gamma \in \kappa\})$, then $A_\alpha \neq (F \cup \bigcup \{\bigcup \{V: V \in \mathcal{U}_\gamma\}: \gamma \in \kappa\})$.

Let $A = \bigcup \{A_\alpha: \alpha \in \kappa^+\}$ and $\mathcal{U} = \{\mathcal{U}_\gamma: 0 < \gamma < \kappa^+\}$. It is clear that $|F \cup A| \leq \lambda^\kappa$. The proof is complete if $X = F \cup A$. Assume, on the contrary, that $p \in X - (F \cup A)$. Since $\psi_c(X) \leq \kappa$, there exists a collection $\{\mathcal{B}_\gamma: \gamma \in \kappa\}$ of open neighborhoods of p in X , such that $\bigcap \{\overline{\mathcal{B}_\gamma}: \gamma \in \kappa\} = \{p\}$. For each $\gamma \in \kappa$, let $U_\gamma = X - \overline{\mathcal{B}_\gamma}$, then $p \notin \overline{U_\gamma}$ and $F \cup A \subseteq \bigcup \{U_\gamma: \gamma \in \kappa\}$. Now, let $\mathcal{W}_\gamma = \{V: V \in \mathcal{B}_x, x \in A \cap U_\gamma \text{ and } V \subseteq U_\gamma\}$; then $\bigcup \mathcal{W}_\gamma \subseteq U_\gamma$, and it is easy to check that $A \cap U_\gamma \subseteq \bigcup \overline{\mathcal{W}_\gamma}$. Since $\text{lc}(X) \leq \kappa$, for each $\gamma \in \kappa$, then there exists $\mathcal{U}_\gamma \in [\mathcal{W}_\gamma]^{\leq \kappa}$ such that $\bigcup \mathcal{W}_\gamma \subseteq F \cup \bigcup \{V: V \in \mathcal{U}_\gamma\}$. Hence, $A \cap U_\gamma \subseteq F \cup \bigcup \{V: V \in \mathcal{U}_\gamma\}$. Since $\mathcal{U}_\gamma \subseteq \mathcal{U}$ and $|\mathcal{U}_\gamma| \leq \kappa$, for all $\gamma \in \kappa$, then by the regularity of κ^+ , there exists $\alpha_0 \in \kappa^+$, such that each $\mathcal{U}_\gamma \in [\mathcal{V}_{\alpha_0}]^{\leq \kappa}$, hence by (3), $A_{\alpha_0} - (F \cup \bigcup \{\bigcup \{V: V \in \mathcal{U}_\gamma\}: \gamma \in \kappa\}) \neq \emptyset$. Since $A_{\alpha_0} \subseteq A \subseteq F \cup \bigcup \{\bigcup \{V: V \in \mathcal{U}_\gamma\}: \gamma \in \kappa\}$, we reach a contradiction. Thus $X = F \cup A$; therefore, $|X| \leq 2^\kappa$. \square

Using Theorem 16, we can derived several corollaries.

Corollary 17. *For $X \in \mathcal{T}_3$, $|X| \leq \pi \chi(X)^{\text{lc}(X)\psi(X)}$.*

Corollary 18. [10] *For $X \in \mathcal{T}_2$, $|X| \leq \pi \chi(X)^{c(X)\psi_c(X)}$.*

Corollary 19. [5] *For $X \in \mathcal{T}_3$, $|X| \leq \pi \chi(X)^{c(X)\psi(X)}$.*

Corollary 20. [4] *For $X \in \mathcal{T}_2$, $|X| \leq 2^{\text{lc}(X)\pi \chi(X)\psi_c(X)}$.*

Corollary 21. *For $X \in \mathcal{T}_3$, $|X| \leq 2^{\text{lc}(X)\pi \chi(X)\psi(X)}$.*

Corollary 22. [5] *For $X \in \mathcal{T}_3$, $|X| \leq 2^{c(X)\pi \chi(X)\psi(X)}$.*

The following example shows that Theorem 16 can give a better estimation than the one in Corollary 18.

Example 23. Let $X = D(2^\kappa)$ be the discrete space of cardinality 2^κ , for some cardinal $\kappa \geq \omega$. Note that $\pi\chi(X) = \psi_c(X) = \omega$, $c(X) = 2^\kappa$, $lc(X) \leq \kappa$; hence $\pi\chi(X)^{c(X)\psi_c(X)} = 2^{2^\kappa} > 2^\kappa = \pi\chi(X)^{lc(X)\psi_c(X)} = |X|$.

Now we turn to the final result of this paper. We shall establish a theorem which strengthens Bell–Ginsburg–Woods’s theorem [3]: For $X \in \mathcal{T}_4$, $|X| \leq 2^{wl(X)\chi(X)}$. Note that except for the added hypothesis that X is normal, the Bell–Ginsburg–Woods’s inequality unifies the two inequalities: For $X \in \mathcal{T}_2$, $|X| \leq 2^{l(X)\chi(X)}$, and for $X \in \mathcal{T}_2$, $|X| \leq 2^{c(X)\chi(X)}$.

Our proof makes use the Pol–Šapirovič’s technique. We refer the reader to [6] for additional inequalities in cardinal functions which can be proved using this technique.

Theorem 24. For $X \in \mathcal{T}_4$, $|X| \leq 2^{qwl(X)\chi(X)}$.

Proof. Let $\kappa = qwl(X)\chi(X)$ and let S be an element of $[X]^{\leq 2^\kappa}$ witnessing that $qwl(X) \leq \kappa$. For each $x \in X$, let \mathcal{B}_x be a local base of x in X such that $|\mathcal{B}_x| \leq \kappa$.

Construct an increasing sequence $\{A_\alpha: \alpha \in \kappa^+\}$ of closed sets in X and a sequence $\{\mathcal{V}_\alpha: \alpha \in \kappa^+\}$ of open collections in X such that

- (1) $|A_\alpha| \leq 2^\kappa$, $0 \leq \alpha < \kappa^+$;
- (2) $\mathcal{V}_\alpha = \bigcup \{\mathcal{B}_x: x \in cl_X(\bigcup_{\beta < \alpha} A_\beta)\}$; $0 < \alpha < \kappa^+$;
- (3) if $\mathcal{U} \subseteq \bigcup \{\mathcal{B}_x: x \in cl_X(\bigcup_{\beta < \alpha} A_\beta)\}$ with $|\mathcal{U}| \leq \kappa$, such that $X - (\overline{S \cup \bigcup \mathcal{U}}) \neq \emptyset$, then $A_\alpha - (\overline{S \cup \bigcup \mathcal{U}}) \neq \emptyset$.

Let $A = \bigcup \{A_\alpha: \alpha \in \kappa^+\}$ and note that A is closed in X ; moreover, clearly $|\overline{S \cup A}| \leq 2^\kappa$. The proof is complete if $X = \overline{S \cup A}$. Suppose not, and let $p \in X - (\overline{S \cup A})$; since X is regular, there exists an open set R such that $\overline{S \cup A} \subseteq R$ and $p \notin \overline{R}$. Let $\mathcal{V} = \{V: V \in \mathcal{B}_p, x \in A, V \subseteq R\}$; and let $G = \bigcup \mathcal{V}$. Clearly $A \subseteq G$ and $p \notin \overline{G}$. Now, since X is normal, there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq G$. Clearly, $\mathcal{V} \cup \{X - \overline{U}\}$ cover X ; so by $qwl(X) \leq \kappa$, there exists $\mathcal{U} \in [\mathcal{V}]^{\leq \kappa}$ such that $X = \overline{S \cup \bigcup \mathcal{U}} \cup X - \overline{U}$. Since $A \cap X - \overline{U} = \emptyset$, $A \subseteq \overline{S \cup \bigcup \mathcal{U}}$. Finally since $|\mathcal{U}| \leq \kappa$, by regularity of κ^+ , there exists $\alpha_0 \in \kappa^+$ such that $\mathcal{U} \subseteq \bigcup \{\mathcal{B}_x: x \in cl_X(\bigcup_{\beta < \alpha_0} A_\beta)\}$. Thus as $X - (\overline{S \cup \bigcup \mathcal{U}}) \neq \emptyset$ then by (3), $A_{\alpha_0} - (\overline{S \cup \bigcup \mathcal{U}}) \neq \emptyset$. A contradiction. \square

Corollary 25 (Bell–Ginsburg–Woods). If X is a T_4 -space, then $|X| \leq 2^{wl(X)\chi(X)}$.

The space X in the Example 23 is such that $|X| = 2^{qwl(X)\chi(X)} < 2^{wl(X)\chi(X)}$. This shows that the Theorem 24 can give a better estimation than the one in Corollary 25.

4. Questions

We present here the list of questions that we could not solve while working on this paper.

Problem 26. If X is a T_1 -space, is $nw(X) \leq psw(X)^{ql(X)}$?

Problem 27. If X is a T_1 -space, is $|X| \leq psw(X)^{ql(X)\psi(X)}$?

Problem 28. If X is a T_2 -space, is $|X| \leq \pi\chi(X)^{ac(X)\psi_c(X)}$?

Problem 29. If X is a T_3 -space, is $|X| \leq 2^{qwl(X)\chi(X)}$?

Problem 30. If X is a T_4 -space, then $|X| \leq 2^{qwl(X)\psi(X)\iota(X)}$?

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